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Energy Decay for Solutions of Ultrahyperbolic Inequalities

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1. INTRODUCTION

Let Δ_y be the Laplacian in $y = (y_1, y_2, ..., y_n)$, $n \geqslant 2$, and let A be a second-order, linear, elliptic operator in $x = (x_1, x_2, ..., x_m)$, $m \geqslant 2$, with coefficients depending on x and y. Consider the ultrahyperbolic operator

$$L = \Delta_y - A, \tag{1.1}$$

on a domain $D \times \Gamma$, where D is a bounded domain in $\mathbb{R}^m$ and Γ is a positive cone in $\mathbb{R}^n$. Protter [4] and Murray and Protter [3] showed that a nontrivial solution $u(x, y)$ of the differential inequality

$$| Lu | \leqslant \Phi_0(y) \, | u | + \Phi_1(y) \, | \nabla_x u | + \Phi_2(y) \, | \nabla_y u |, \tag{1.2}$$

such that u vanishes on the boundary of $D \times \Gamma$, cannot tend to zero arbitrarily fast as $| y | \to \infty$. More precisely, they proved in the cases $F(r) = r^2$ and $F(r) = \log r$ that under appropriate conditions on the coefficients in (1.2), the energy associated with nontrivial solutions cannot tend to zero faster than every positive power of $\exp(-F(| y |))$ as $| y | \to \infty$. Recently, Murray [2] has obtained a generalization in the case $F(r) = r^\alpha$, $\alpha > 1$, to ultrahyperbolic operators of the form $B_y - A_x$ where B_y is an elliptic operator in y which is close to Δ_y for large $| y |$.

Our purpose is to extend the results of [3] to more general exponents $F(r)$. In Section 4 we will establish conditions for the "exponential-like" behavior associated with a class of exponents which include $F(r) = r^\alpha$, $\alpha > 0$. Section 5 contains a set of conditions for polynomial decay that are somewhat weaker than those of [3]. Our method of finding conditions for these "maximal" rates of decay is similar to that used in [2, 3, 4] in that a lower bound is established for a certain weighted integral from which an energy lower bound may be deduced. Instead of employing integral identities, however, we will make use of the techniques of ordinary differential inequalities that we applied previously [5, 6], to hyperbolic inequalities.

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2. General Assumptions

Let the ultrahyperbolic operator $L = \Delta_y - A$ be defined on $D \times \Gamma$ where D is a bounded domain in $\mathbb{R}^m$ and Γ is a positive cone in $\mathbb{R}^n$ defined as follows. Suppose C is a domain on the unit sphere in $\mathbb{R}^n$ such that C has a piecewise smooth boundary ∂C and set

$$\Gamma = \{y \in \mathbb{R}^n : y/r \in C,\ r \geqslant 1\},$$

where $r = | y | = (y_1 + \cdots + y_n)^{1/2}$. Also set

$$C(R) = \{y \in \Gamma : | y | = R\}$$

for any $R \geqslant 1$.

We assume the operator A is given in the self-adjoint form

$$A = \sum_{i,j=1}^{m} \frac{\partial}{\partial x_j} \left(a_{ij}(x, y) \frac{\partial}{\partial x_i}\right)$$

with coefficient $a_{ij} = a_{ji}$ of class $C^1(D \times \Gamma)$. Furthermore, we suppose there is a positive function $m(r)$, $r \geqslant 1$, and a positive constant δ such that

$$\sum_{i,j=1}^{m} a_{ij}(x, y)\, \xi_i \xi_j \geqslant m(r)\, | \xi |^2 \tag{2.1}$$

and

$$\sum_{i,j=1}^{m} \frac{\partial a_{ij}}{\partial r} (x, y)\, \xi_i \xi_j \geqslant -(2 - \delta)\, r^{-1} m(r)\, | \xi |^2, \tag{2.2}$$

for all $(x, y) \in D \times \Gamma$ and all $\xi \in \mathbb{R}^m$.

Murray and Protter measured the decay of solutions in terms of the energy integral

$$\mathscr{E}(u; r) = r^{-n+1} \int_{C(r)} (\| u \|^2 + \| \nabla_x u \|^2 + \| u_r \|^2)\, d\sigma,$$

where $\| v \|$ denotes the $L^2(D)$ norm of $| v |$, and $| v |$ is the Euclidean norm of v if $v \in \mathbb{R}^m$. We will state our results in terms of the integral

$$E(u; r) = r^{-n+1} \int_{C(r)} (\| u \|^2 + a(u, u) + \| u_r \|^2)\, d\sigma, \tag{2.3}$$

where $a(v, w)$ is the symmetric, bilinear form

$$a(v, w) = \int_D \sum_{i,j=1}^{m} a_{ij}(x, y)\, v_{x_i} w_{x_j}\, dx. \tag{2.4}$$

The ellipticity condition (2.1) implies that for any function $v \in C^1(D)$,

$$a(v, v) \geqslant m(r) \| \nabla_x v \|^2. \tag{2.5}$$

Similarly, if we define

$$a_r(v, w) = \int_D \sum_{i,j=1}^{m} \frac{\partial a_{ij}}{\partial r} (x, y)\, v_{x_i} w_{x_j}\, dx,$$

then, as a consequence of (2.2), we have

$$a_r(v, v) \geqslant -(2 - \delta)\, r^{-1} m(r) \| \nabla_x v \|^2. \tag{2.6}$$

3. The Weighted Integral

The weights are positive functions of r expressed in the form $\exp(\lambda F(r))$, where λ is a positive parameter and F is a nonnegative, increasing function of class $C^3[1, \infty)$. Further assumptions on F will be made in Sections 4 and 5. Here, we will derive an expression for the derivative of a weighted integral involving any function $u \in C^2(D \times \Gamma)$ which satisfies the boundary condition

$$u(x, y) = 0 \qquad \text{if} \quad x \in \partial D \quad \text{or} \quad y/r \in \partial C. \tag{3.1}$$

If we set $w = u \exp(\lambda F(r))$ and $f = F'$, then

$$\Delta_y u = e^{-\lambda F}(\Delta_y w - 2\lambda f w_r + r^{-2} h w), \tag{3.2}$$

where

$$h(r, \lambda) = \lambda r[\lambda r f^2(r) - (n - 1) f(r) - r f'(r)]. \tag{3.3}$$

Now define

$$Q(r, \lambda) = r^{-n+1-\mu} \int_{C(r)} [r^2 \| w_r \|^2 - \| \nabla_S w \|^2 + r^2 a(w, w) + h \| w \|^2]\, d\sigma, \tag{3.4}$$

where μ is nonnegative constant depending on the function F. The symbol ∇_S denotes the gradient operator on the unit sphere S_{n-1} in $\mathbb{R}^n$, defined [1] as

$$\nabla_S = (d_1, d_2, \ldots, d_n),$$

with d_k operating on functions $v \in C^1(S_{n-1})$ by

$$d_k v(\eta) = \left[\frac{\partial}{\partial y_k} v(y/r)\right]_{y=\eta}.$$

In this notation, the Laplace–Beltrami operator on S_{n-1} is given by

$$\Delta_S = \sum_{k=1}^{n} d_k^2$$

and the Laplacian in $\mathbb{R}^n$ by

$$\Delta_y = \partial^2/\partial r^2 + (n-1)\, r^{-1}\partial/\partial r + r^{-2}\Delta_S\,.$$

THEOREM 1. *Let* $u \in C^2(D \times \Gamma)$ *satisfy the boundary condition* (3.1). *If* $w = u \exp(\lambda F(r))$, *then the derivative of the integral* (3.4) *with respect to* r *is given by*

$$\begin{aligned} Q_r(r, \lambda) = r^{-\mu} \int_C [&2r^2 e^{\lambda F}(w_r\,, Lu) + r(4\lambda r f - 2n + 4 - \mu)\, \| w_r \|^2 \\ &+ \mu r^{-1} \| \nabla_S w \|^2 + (2-\mu)\, r a(w, w) \\ &+ r^2 a_r(w, w) + (h_r - \mu r^{-1} h)\, \| w \|^2]\, d\sigma, \end{aligned} \tag{3.5}$$

where $(\cdot, \cdot)$ *denotes the inner product in* $L^2(D)$.

Proof. Introducing spherical coordinates $y = (r, \eta)$ into the integral (3.4), we may write

$$Q(r, \lambda) = r^{-\mu} \int_C [r^2 \| w_r \|^2 - \| \nabla_S w \|^2 + r^2 a(w, w) + h \| w \|^2]\, d\sigma.$$

Now differentiate with respect to r to obtain

$$\begin{aligned} Q_r(r, \lambda) = r^{-\mu} \int_C [&2r^2(w_r\,, w_{rr}) - 2(\nabla_S w, \nabla_S w_r) + 2r^2 a(w, w_r) \\ &+ 2h(w, w_r) + r^2 a_r(w, w) + (2-\mu) r \| w_r \|^2 \\ &+ \mu r^{-1} \| \nabla_S w \|^2 + (2-\mu)\, r a(w, w) \\ &+ (h_r - \mu r^{-1} h)\, \| w \|^2]\, d\sigma. \end{aligned} \tag{3.6}$$

Integrating by parts over C in the second term and over D in the third term of (3.6) and making use of the boundary condition (3.1), we may combine the first four terms of the integrand in (3.6) into

$$\begin{aligned} &2r^2(w_r\,, w_{rr} + r^{-2}\Delta_S w - Aw + r^{-2} h w) \\ &\quad = 2r^2(w_r\,, Lw + r^{-2} h w) - 2(n-1) r \| w_r \|^2 \\ &\quad = 2r^2 e^{\lambda F}(w_r\,, Lu) + 2r(2\lambda r f - n + 1) \| w_r \|^2, \end{aligned}$$

where we have made use of the identity (3.2) to obtain the last expression. Substitution of this into (3.6) yields (3.5).

An immediate corollary of Theorem 1, by virtue of the inequalities (2.5) and (2.6), is the lower bound

$$Q_r(r, \lambda) \geqslant r^{-\mu} \int_C [2r^2 e^{\lambda F}(w_r, Lu) + r(4\lambda r f - 2n + 4 - \mu) \| w_r \|^2 + \mu r^{-1} \| \nabla_S w \|^2 + (\delta - \mu) rm \| \nabla_x w \|^2 + (h_r - \mu r^{-1} h) \| w \|^2] \, d\sigma. \tag{3.7}$$

4. Exponential-Like Decay

In this section we study energy decay associated with the class of exponents F whose first derivatives f are positive, twice continuously differentiable functions satisfying the additional conditions:

$$(rf)' \geqslant c_0 f \quad \text{for some constant} \quad c_0 > 0, \tag{4.1a}$$

$$f' = O(f^2) \quad \text{as} \quad r \to \infty, \tag{4.1b}$$

$$rf'' = O(f^2) \quad \text{as} \quad r \to \infty. \tag{4.1c}$$

We refer to the corresponding asymptotic behavior as "exponential-like" since the functions $F(r) = r^\alpha$, $\alpha > 0$, are members of this class. The conditions (4.1) are used to obtain estimates for $\log r$ and $\log f$ and establish specific bounds on h and h_r.

Lemma 1. *If f satisfies the conditions* (4.1a) *and* (4.1b), *then*

$$\log r = O(F) \quad \textit{and} \quad \log f = O(F) \quad \textit{as} \quad r \to \infty.$$

Proof. Note that (4.1a) implies $rf(r) \geqslant f(1) > 0$ for $r \geqslant 1$. Thus, as a result of (4.1a) and (4.1b), both assertions are simple consequences of the fact that the logarithmic derivatives of r and f are bounded by a constant multiple of $f = F'$.

Lemma 2. *If f satisfies the conditions* (4.1), *then there is a constant $\lambda_0 > 0$ such that if $\lambda \geqslant \lambda_0$*

$$\tfrac{1}{2}[\lambda r f(r)]^2 \leqslant h(r, \lambda) \leqslant 2[\lambda r f(r)]^2 \tag{4.2}$$

and

$$h_r(r, \lambda) \geqslant c_0 \lambda^2 r f^2(r) \tag{4.3}$$

for all $r \geqslant 1$.

Proof. The estimates (4.2) follow easily from (4.1a) and (4.1b) if we express h in the form

$$h(r, \lambda) = (\lambda r f)^2 \{1 - \lambda^{-1}[(n - 1)/(rf) + f'/f^2]\}.$$

Similarly, the lower bound (4.3) is an immediate consequence of the conditions (4.1), since

$$h_r(r, \lambda) = \lambda^2 r f^2\{2(rf)'/f - \lambda^{-1}[rf''/f^2 + (n+1)\, f'/f^2 + (n-1)/(rf)]\}.$$

Note that λ_0 depends only on n, $f(1)$, and the constants in (4.1).

Next, we establish the basic lower bound which, together with the assumptions on f, will allow us to specify conditions for the energy associated with a nontrivial solution to decay no faster than a positive power of $\exp(-F(r))$.

LEMMA 3. *If $u \in C^2(D \times \Gamma)$ is a solution of*

$$\| Lu \|^2 \leqslant \varphi_0(r)\, \| u \|^2 + \varphi_1(r)\, \| \nabla_x u \|^2 + \varphi_2(r)\, \| \nabla_y u \|^2 \tag{4.4}$$

and satisfies the boundary condition (3.1), *then*

$$\begin{aligned} Q_r(r, \lambda) \geqslant r^{-\mu} \int_C &\{r[3\lambda r f - 2n + 4 - \mu - 2r\varphi_2/(\lambda f)]\, \| w_r \|^2 \\ &+ [\mu r^{-1} - \varphi_2/(\lambda f)]\, \| \nabla_S w \|^2 + [(\delta - \mu)\, rm - r^2\varphi_1/(\lambda f)]\, \| \nabla_x w \|^2 \\ &+ [h_r - \mu r^{-1} h - r^2\varphi_0/(\lambda f) - 2\lambda r^2 f \varphi_2]\, \| w \|^2\}\, d\sigma. \end{aligned} \tag{4.5}$$

Proof. For any positive function $\theta(r, \lambda)$, we have

$$2e^{\lambda F}(w_r, Lu) \geqslant -\theta^{-1}\, \| w_r \|^2 - \theta e^{2\lambda F}\, \| Lu \|^2.$$

Applying (4.4) to this inequality and using the relation

$$\| \nabla_y u \|^2 = \| u_r \|^2 + r^{-2}\, \| \nabla_S u \|^2,$$

we obtain

$$\begin{aligned} 2e^{\lambda F}(w_r, Lu) &\geqslant -\theta^{-1}\, \| w_r \|^2 - \theta[\varphi_0\, \| w \|^2 + \varphi_1\, \| \nabla_x w \|^2 \\ &\qquad + \varphi_2(\| w_r - \lambda f w \|^2 + r^{-2}\, \| \nabla_S w \|^2)] \\ &\geqslant -(\theta^{-1} + 2\theta\varphi_2)\, \| w_r \|^2 - r^{-2}\theta\varphi_2\, \| \nabla_S w \|^2 \\ &\qquad - \theta\varphi_1\, \| \nabla_x w \|^2 - \theta(\varphi_0 + 2\lambda^2 f^2 \varphi_2)\, \| w \|^2. \end{aligned}$$

Using this inequality in (3.7), we find that

$$\begin{aligned} Q_r(r, \lambda) \geqslant r^{-\mu} \int_C &\{r(4\lambda r f - 2n + 4 - \mu - r\theta^{-1} - 2r\theta\varphi_2)\, \| w_r \|^2 \\ &+ (\mu r^{-1} - \theta\varphi_2)\, \| \nabla_S w \|^2 + [(\delta - \mu)\, rm - r^2\theta\varphi_1]\, \| \nabla_x w \|^2 \\ &+ [h_r - \mu r^{-1} h - \theta r^2(\varphi_0 + 2\lambda^2 f^2 \varphi_2)]\, \| w \|^2\}\, d\sigma. \end{aligned}$$

If we choose $\theta = (\lambda f)^{-1}$, then (4.5) is the result.

THEOREM 2. *Let* $u \in C^2(D \times \Gamma)$ *satisfy the differential inequality* (4.4) *on* $D \times \Gamma$ *and the boundary condition* (3.1). *Suppose* f *satisfies the conditions* (4.1) *and assume*

$$\textit{either} \quad \varphi_0 = O(r^{-1}f^3) \quad \textit{or} \quad O(r^{-1}mf) \qquad \textit{as} \quad r \to \infty, \tag{4.6a}$$

$$\varphi_1 = O(r^{-1}mf) \qquad \textit{as} \quad r \to \infty, \tag{4.6b}$$

$$\varphi_2 = O(r^{-1}f) \qquad \textit{as} \quad r \to \infty. \tag{4.6c}$$

If

$$\lim_{r\to\infty} e^{\beta F(r)}\, E(u; r) = 0 \tag{4.7}$$

for every $\beta > 0$, *then* $u \equiv 0$.

Proof. We first show that, if $0 < \mu < \min(\delta, c_0/2)$, there is a $\lambda_1 \geqslant \lambda_0$ such that $Q_r(r, \lambda) \geqslant 0$ for all $r \geqslant 1$ and all $\lambda \geqslant \lambda_1$. Note that (4.6c) implies that $\varphi_2 = O(f^2)$ as $r \to \infty$ since $rf(r) \geqslant f(1) > 0$. Thus the fact that the coefficients of $\| w_r \|^2$, $\| \nabla_x w \|^2$, and $\| \nabla_S w \|^2$ in (4.5) are positive for λ sufficiently large is an immediate consequence of the conditions (4.6). The coefficient of $\| w \|^2$ in (4.5) may also be seen to be positive for large λ under the first of conditions (4.6a) and condition (4.6c), since by Lemma 2,

$$h_r - \mu r^{-1} h \geqslant (c_0 - 2\mu)\, \lambda^2 r f^2$$

for $\lambda \geqslant \lambda_0$. If $f^2 = o(m)$ as $r \to \infty$, then we may obtain the positivity of the coefficient of $\| w \|^2$ under the (then) weaker second condition of (4.6a). This follows from the fact that, due to the boundary condition (3.1), there is a constant K depending only on the size of D such that

$$\| w \|^2 \leqslant K \| \nabla_x w \|^2. \tag{4.8}$$

Note that λ_1 depends only on n, $f(1)$, and the constants in (4.1) and (4.6).

Using the upper bound (4.2) for h, we find from (3.4) that

$$Q(r, \lambda) \leqslant r^{-n+3-\mu} e^{2\lambda F} \int_{C(r)} [\| u_r + \lambda f u \|^2 + a(u, u) + 2\lambda^2 f^2 \| u \|^2]\, d\sigma$$

for $\lambda \geqslant \lambda_0$. Thus the estimates of Lemma 1 may be applied to give us the bound

$$Q(r, \lambda) \leqslant \text{const} \cdot \lambda^2 e^{2(\lambda + c)F} E(u; r) \tag{4.9}$$

for some constant c depending on f and for all $r \geqslant 1$ and all $\lambda \geqslant \lambda_0$. Now fix $r_0 \geqslant 1$. Then, since Q is a nondecreasing function of r for $\lambda \geqslant \lambda_1$, we have

$$Q(r_0, \lambda) \leqslant Q(r, \lambda) \tag{4.10}$$

for $r \geqslant r_0$ and $\lambda \geqslant \lambda_1$. Thus, if $\lambda \geqslant \lambda_1$ and $r \geqslant r_0$, both (4.9) and (4.10) hold. But the hypothesis (4.7) implies that the right side of (4.9) tends to zero as $r \to \infty$. It follows that $Q(r_0, \lambda) \leqslant 0$. On the other hand, using the lower bound (4.2) for h, we find that

$$Q(r_0, \lambda) \geqslant r_0^{-n+3-\mu} e^{2\lambda F(r_0)} \int_{C(r_0)} [-\| \nabla_S u \|^2 + a(u, u) + \tfrac{1}{2}\lambda^2 f^2 \| u \|^2] \, d\sigma$$

for $\lambda \geqslant \lambda_0$. Hence, if $u \not\equiv 0$ for $| y | = r_0$, then $Q(r_0, \lambda) > 0$ for λ sufficiently large. Thus $u \equiv 0$ for $| y | = r_0$. Since r_0 is arbitrary, u must vanish identically.

Remarks. 1. If $F(r) = r^\alpha$, $\alpha > 0$, and $m(r) = m_0$, a positive constant, the conditions (4.6) become (i) either $\varphi_0 = O(r^{3\alpha-4})$ or $O(r^{\alpha-2})$ as $r \to \infty$, (ii) $\varphi_1 = O(r^{\alpha-2})$ as $r \to \infty$, and (iii) $\varphi_2 = O(r^{\alpha-2})$ as $r \to \infty$. Note that these include the condition, found in [3], that φ_0, φ_1, and φ_2 be uniformly bounded in the case $\alpha = 2$. However, the first of the conditions (i), $\varphi_0 = O(r^2)$, provides a better result in this case.

2. We should mention that although one sometimes refers to this type of result as determining a "maximal" decay rate for a class of solutions, we make no assertion about the existence of a maximal rate of decay nor, given its existence, what that maximal rate is. In special cases, however, it is possible to find functions $u(x, y)$ which exhibit this type of behavior. For example, let ξ and θ be eigenfunctions of the Dirichlet problem for Δ_x on D and Δ_S on C, respectively, and let $\rho(r) = \exp(-r^\alpha)$ with $\alpha > 0$. Then the function $u(x, y) = \xi(x)\, \theta(\eta)\, \rho(r)$ has energy $E(u; r) = O(\exp(-\gamma r^\alpha))$ as $r \to \infty$ for some constant $\gamma > 0$ and u is a nontrivial solution of

$$\| \Delta_y u - \Delta_x u \|^2 \leqslant \varphi_0(r) \| u \|^2,$$

with $\varphi_0(r) = O(r^{4\alpha-4})$ as $r \to \infty$. But we can only conclude from the theorem that $E(u; r)$ cannot tend to zero faster than every positive power of $\exp(-r^{4\alpha/3})$.

5. Polynomial Decay

The results of the preceding section cannot be applied to determine conditions for polynomially bounded decay. The difficulty arises from the fact that the function $F(r) = \log r$ does not satisfy condition (4.1a) and this in turn means that h_r does not satisfy the lower bound (4.3). Indeed, it is easy to verify that $h = \lambda^2 - (n-2)\lambda$, so (4.2) is valid but $h_r \equiv 0$ in this case. Murray and Protter overcame the difficulty by assuming $\varphi_2 \equiv 0$ in the differential inequality (4.4) and using the inequality (4.8). By picking a slightly different function, namely $F(r) = \log(r + 1)$, which still gives polynomially bounded decay, we will demonstrate that a somewhat more general result is valid. Although the corresponding function h still does not satisfy (4.3), we will prove that h_r is bounded below by a positive function for sufficiently large λ.

LEMMA 4. *If* $F(r) = \log(r + 1)$, *then there is a* $\lambda_0 > 0$ *such that if* $\lambda \geqslant \lambda_0$ *then* h *satisfies* (4.2) *and*

$$h_r(r, \lambda) \geqslant \lambda^2 r(r + 1)^{-3} \tag{5.1}$$

for all $r \geqslant 1$.

Proof. The assertions follow from

$$h(r, \lambda) = \lambda^2 r^2 (r + 1)^{-2}\{1 - \lambda^{-1}[(n - 1)(1 + 1/r) - 1]\}$$

and

$$h_r(r, \lambda) = \lambda^2 r(r + 1)^{-3}\{2 - \lambda^{-1}[(n - 1)(1 + 1/r) - 2]\}.$$

In this case, λ_0 depends only on n.

THEOREM 3. *Let* $u \in C^2(D \times \Gamma)$ *satisfy the differential inequality*

$$\| Lu \|^2 \leqslant \varphi_0(r) \| u \|^2 + \varphi_1(r) \| \nabla_x u \|^2 + \varphi_3(r) \| u_r \|^2 \tag{5.2}$$

and the boundary condition (3.1). *Suppose*

$$\text{either} \quad \varphi_0 = O(r^{-5}) \quad \text{or} \quad \varphi_0 = O(r^{-2}m) \qquad \text{as} \quad r \to \infty, \tag{5.3a}$$

$$\varphi_1 = O(r^{-2}m) \qquad \text{as} \quad r \to \infty, \tag{5.3b}$$

$$\varphi_3 = O(r^{-3}) \qquad \text{as} \quad r \to \infty. \tag{5.3c}$$

If

$$\lim_{r \to \infty} r^\beta E(u; r) = 0$$

for every $\beta > 0$, *then* $u \equiv 0$.

Proof. As in Lemma 3, we begin by applying the differential inequality (5.2) to the inequality (3.7) for Q_r. Here, however, we set $f(r) = (r + 1)^{-1}$ and $\theta = \lambda^{-1}(r + 1)$. Moreover, there will be no term in $\| \nabla_S w \|^2$ since we have replaced $\| \nabla_y u \|^2$ by $\| u_r \|^2$ in the differential inequality and taken $\mu = 0$. Thus, in place of the inequality (4.5), we have

$$\begin{aligned} Q_r(r, \lambda) \geqslant \int_C \{ & r[3\lambda r(r + 1)^{-1} - 2n + 4 - 2\lambda^{-1} r(r + 1)\varphi_3] \| w_r \|^2 \\ & + [\delta r m - \lambda^{-1} r^2 (r + 1)\varphi_1] \| \nabla_x w \|^2 \\ & + [h_r - \lambda^{-1} r^2 (r + 1)\varphi_0 - \lambda r^2 (r + 1)^{-1}\varphi_3] \| w \|^2 \} \, d\sigma. \end{aligned} \tag{5.4}$$

The hypotheses (5.3) on φ_0, φ_1, and φ_3 together with the lower bound on h_r of Lemma 4 imply that $Q_r \geqslant 0$ for λ sufficiently large. As before, we apply the second of conditions (5.3a) if we use (4.8) to incorporate the term in φ_0 of (5.4) into the coefficient of $\| \nabla_x w \|^2$. We omit the remainder of the proof, since it proceeds exactly as in Theorem 2.

In the case $m(r) = m_0 > 0$, the conditions (5.3) become $\varphi_0 = O(r^{-2})$, $\varphi_1 = O(r^{-2})$, and $\varphi_3 = O(r^{-3})$ as $r \to \infty$, which include the results obtained in [3].

References

1. S. Agmon, Unicité et convexité dans les problèmes différentiels, *in* "Sem. Math. Sup. (1965)," Univ. of Montreal Press (1966).
2. A. C. Murray, Asymptotic behavior and uniqueness for an ultrahyperbolic equation with variable coefficients, *J. Differential Equations* **23** (1977), 200–215.
3. A. C. Murray and M. H. Protter, Asymptotic behavior and the Cauchy problem for ultrahyperbolic operators, *Indiana Univ. Math. J.* **24** (1974), 115–130.
4. M. H. Protter, Asymptotic decay for ultrahyperbolic operators, *in* "Contributions to Analysis," Academic Press, New York, 1974.
5. H. Ogawa, Lower bounds for solutions of hyperbolic inequalities, *Proc. Amer. Math. Soc.* **16** (1965), 853–857.
6. H. Ogawa, Lower bounds for solutions of hyperbolic inequalities on expanding domains, *J. Differential Equations* **13** (1973), 384–390.